CONTROLLABILITY VIA AN APPROXIMATION PROBLEM

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Abstract
This paper is concerned with a nonlinear time-discrete dynamical system whose dynamics is described by a system of vector difference equations involving state and control vector functions. It can be seen as a contribution to the investigation of problems of controllability via the solution of an approximation problem. The motivation comes from an actual interdisciplinary research field in the area of environmental systems [4]. The special structure of the developed TEM-model permits two transformations which lead to a solution of the problem of controllability within the smallest number of time-steps, if the problem is solvable [3]. Founded upon these results, the presented algorithm can be determined.

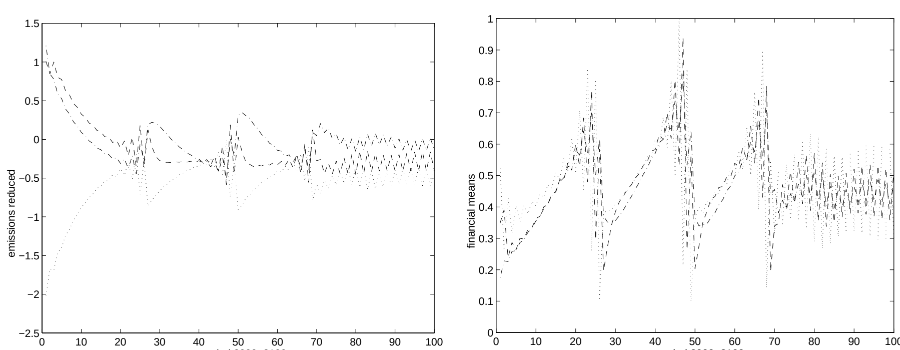
Introduction

The TEM-model (Technology-Emission-Means-model) is a non-linear time-discrete model which describes the economical interaction between several actors. These actors intend to optimize their objective function E_i (reduced emissions which are caused by technologies T_i) by means of expenditures of money or financial means M_i, respectively. The index stands for the i-th player $i = 1, \ldots, n$. The TEM-model was developed to simulate an economic Joint-Implementation Program. Such a process plays a central role in order to fulfil the environment treaties of Rio or Kyoto, respectively. Simulating this economic situation, the actors are linked by technical cooperations and the market. This behavior is expressed by the parameter of the em-matrix which is the central element of the TEM-model:

$$\begin{aligned} E_i(t+1) &= E_i(t) + \sum_{j=1}^{n} em_{ij}(t) M_j(t) \\ M_i(t+1) &= M_i(t) - \lambda_i M_i(t)[M_i^* - M_i(t)]\{E_i(t) + \varphi_i \Delta E_i(t)\} \end{aligned} \tag{1}$$

E_i	reduced emissions of actor i
M_i	financial means of actor i
λ_i	growth parameter
φ_i	memory parameter

The em_{ij}-parameter describes the effect on the emissions of the i-th actor, if the j-th actor invests money. We can say that it expresses how effective technological cooperations are, which is the kernel of a Joint-Implementation Program. If we let $em_{ij}(t) = em^*_{ij}, t = 0, \ldots, N$, i.e. the economic relationships are constant over a long period, we are able to determine the fixed points of the dynamical system and they are not attractive, even chaotic behaviour is observable:



For a detailed analysis of the TEM-model see [4].

actor	emissions	means	budget	φ	λ	em-matrix		
1	-1	0.3	1	11	0.82	1	-0.7	-0.3
2	0.6	0.1	1	11	0.25	-0.8	1	-0.2
3	0.5	0.2	1	11	0.4	-0.9	-0.1	1

data for observing chaos

In order to reach these steady states, an independent institution may influence the trade relations between the actors. Mathematically, the control parameters have to be determined:

The Control Problem

Let us represent the time-discrete dynamical system in (1) by general difference equations added with control vector functions of the form:

$$
\begin{gathered}
x_i(t+1) = x_i(t) + f_i(x(t), u(t)) \\
x(t) = (x_1(t), \ldots, x_n(t)) \quad u(t) = (u_1(t), \ldots, u_n(t))
\end{gathered}
\tag{2}
$$

for $i = 1, \ldots, n$ and $t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here $x_i : \mathbb{N}_0 \to \mathbb{R}^{l_i}$ and $u_i : \mathbb{N}_0 \to \mathbb{R}^{m_i}$ for $i = 1, \ldots, n$ are state and control vector functions, respectively. Furthermore, $f_i : \prod_{j=1}^{n} \mathbb{R}^{l_j} \times \prod_{j=1}^{n} \mathbb{R}^{m_j} \to \mathbb{R}^{l_j}, i = 1, \ldots, n$ are given vector functions. In addition we assume, for every $i = 1, \ldots, n$, non-empty sets $X_i \subseteq \mathbb{R}^{l_i}$ and $U_i \subseteq \mathbb{R}^{m_i}$ to be given and require control conditions of the form $u_i(t) \in U_i$ for all $i = 1, \ldots, n$ and $t \in \mathbb{N}_0$ as well as state constraints of the form $x_i(t) \in X_i$ for all $i = 1, \ldots, n$ and $t \in \mathbb{N}_0$. We assume $X_i = \mathbb{R}^{l_i}$ for $i = 1, \ldots, n$ to hold and choose some $N \in \mathbb{N}$.

Then we consider the following **approximation problem**:
Find control functions $u_i : \mathbb{N}_0 \to \mathbb{R}^{m_i}$ with $u_i(t) \in U_i$ for $t = 0, \ldots, N-1$ and $i = 1, \ldots, n$ such that under the conditions

$$
x_i(t+1) = x_i(t) + f_i(x(t), u(t)), \quad t = 0, \ldots, N-1 \quad \text{and} \quad x_i(0) = x_{0i}
$$

for $i = 1, \ldots, n$ the function value

$$
\varphi_N(u) = \sum_{j=1}^{n} (\, \|x_i(N) - \widehat{x}_i\|_2^2 + \|u_i(N-1)\,\|_2^2)
$$

is as small as possible. If the problem of controllability has a solution, then there is some $N \in \mathbb{N}$ such that for every solution of the above problem it necessarily follows that $u_i(N-1) = \Theta_{m_i}$ and $x_i(N) = \widehat{x}_i$ for $i = 1, \ldots, n$, Θ_{m_i} is the zero vector of $\mathbb{R}^{m_i}$. Hence by solving the above problem, one also obtains a solution of the problem of controllability. The solution of the above problem can be achieved with the help of an algorithm [3]:

The algorithm

We choose control functions $u_i^0 : \mathbb{N}_0 \to \mathbb{R}^{m_i}$ with

$$
u_i^0(t) \in U_i \quad \text{for} \quad t = 0, \ldots, N-1 \quad \text{and} \quad i = 1, \ldots, n
$$

$$
(\text{for instance } u_i^0(t) = \Theta_{m_i} \quad \text{for} \quad t = 0, \ldots, N-1 \quad \text{and} \quad i = 1, \ldots, n)
$$

and calculate

$$
x_i^0(t+1) = x_i^0(t) + f_i(x^0(t), u^0(t))
$$

for $t = 0, \ldots, N-1$ with $x_i^0(0) = x_{0i}$ for $i = 1, \ldots, n$.
Then we construct a sequence

$$
(u^k)_{k \in \mathbb{N}_0} \quad \text{in} \quad \left\{ u : \{0, \ldots, N-1\} \to \prod_{j=1}^{n} U_j \right\}
$$

and a sequence

$$(x^k)_{k\in\mathbb{N}_0} \quad \text{in} \quad \left\{ x : \{0,\dots,N\} \to \prod_{j=1}^{n} \mathbb{R}^{l_j} \right\}$$

as follows: If u^k and x^k are given for some $k \in \mathbb{N}_0$, then we determine

$$u_i^{k+1}(t) \in U_i \quad \text{for} \quad t = 0,\dots,N-1 \quad \text{and} \quad i = 1,\dots,n$$

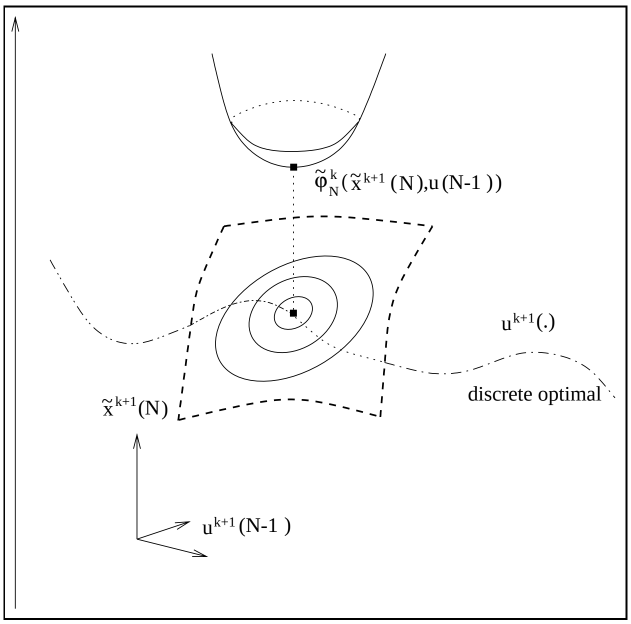
such that under the conditions

$$\tilde{x}^{k+1}(t+1) = \tilde{x}^{k+1}(t) + f(x^k(t), u^{k+1}(t)) \quad \text{for} \quad t = 0,\dots,N-1$$

and $\tilde{x}^{k+1}(0) = x_0$ the (modified) function value

$$\varphi_N^k(u^{k+1}) \quad = \quad \sum_{i=1}^{n} (\|\tilde{x}_i^{k+1}(N) - \hat{x}_i\|_2^2 + \|u_i^{k+1}(N-1)\|_2^2) \tag{3}$$

becomes minimal. The following figure may reflect this:



Now we obtain the following transformed objective function taking advantage of the special structure of the discrete dynamics which can also be illustrated on the next page.

$$\varphi_N^k(u^{k+1}) \quad = \quad \sum_{i=1}^{n} (\| \sum_{t=0}^{N-1} f_i(x^k(t), u^{k+1}(t)) + x_{0i} - \hat{x}_i\|_2^2 + \|u_i^{k+1}(N-1)\|_2^2)$$

If $u^{k+1}(t)$ has been determined for $t = 0,\dots,N-1$, then we calculate

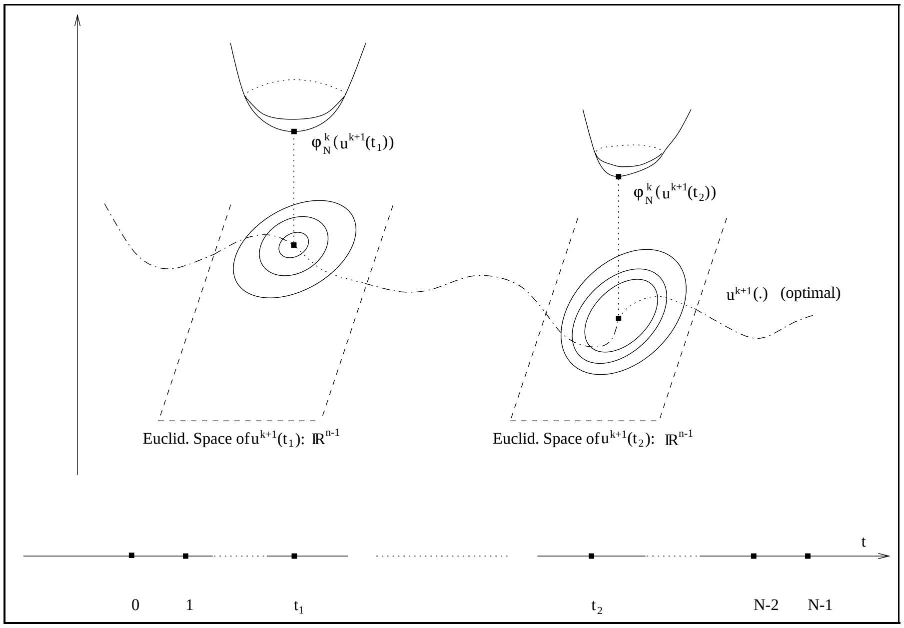
$$x^{k+1}(t+1) = x^{k+1}(t) + f(x^{k+1}(t), u^{k+1}(t))$$

$$\text{for} \quad t = 0,\dots,N-1 \quad \text{where} \quad x^{k+1}(0) = x_0.$$

If $x^{k+1}(t) = \tilde{x}^{k+1}(t)$ for all $t = 0,\dots,N$, then we have found a **solution** of the above problem. Otherwise we proceed with u^{k+1} and x^{k+1} instead of u^k and x^k, respectively. Let us make the assumption that all functions

$$f_i : \prod_{j=1}^{n} \mathbb{R}^{l_j} \times \prod_{j=1}^{n} \mathbb{R}^{m_j} \to \mathbb{R}^{l_i}$$

for $i = 1,\dots,n$ are continuous. Then we have the following



Theorem 1 *If for every $t \in \{0, \ldots, N-1\}$, there is some*

$$u(t) \in \prod_{j=1}^{n} U_j \quad with \quad u(t) = \lim_{k \to \infty} u^k(t)$$

then $u_i(t)$ for $t = 0, \ldots, N-1$ and $i = 1, \ldots, n$ solve the above problem.

Conclusion

The dynamics of the so called TEM-model decribe an environmental system which contains additionally a technical dimension. In order to reach steady states of the TEM-model which are comparable to the CO_2-values mentioned in the Kyoto protocol the problem of controllability has to be formulated. Via the solution of an approximation problem the problem of controllability can be solved. Taking advantage of the special structure of the discrete dynamics a new algorithm based on a proved existence theorem can be determined.

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